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Boltzmann Equation with Fluctuations

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Abstract

From the Liouville equation, by the method of multiple-time-scales, a generalized Boltzmann equation with fluctuations is obtained on the statistical considerations of the randomness of the many-particle correlations in the macroscopic picture. These fluctuations lead to an H theorem in which the H function decreases, with fluctuations, with time toward equilibrium. These fluctuations furnish a source for a random force term introduced by Fox and Uhlenbeck in the Boltzmann equation.

1. *Introduction*

The following properties of the Boltzmann equation are well known:

(1) As a consequence of the Stosszahlansatz in the formulation of the collision integral, the Boltzmann equation is not invariant upon the reversal of time, and the time arrow is expressed by the H theorem

$$\frac{dH}{dt} \leqslant 0 \tag{1.1}$$

(2) The monotonic decrease of the H function shows that the Boltzmann equation describes a monotonic approach to equilibrium without fluctuations.

On the other hand, in the attempts by Boltzmann (Boltzmann, 1898; Ehrenfest, 1911) to reinterpret the H theorem in view of the criticisms of Loschmidt and Zermelo, the probability basis of the H theorem was emphasized, namely, when the state of the gas is not that of equilibrium, the collision integral is to represent the overwhelmingly large probability that the collision leads to a decrease in the value of the H function. From the collision integral and the definition of the H function, it follows that this probability of decrease of H remains the same if the velocities of the two colliding particles are reversed.† According to this reformulation of the meaning of the H theorem, the H function can rise above the minimum

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† But this does not mean that H decreases both in the forward and the backward directions of time.

(or, equilibrium) value given by $dH/dt = 0$, only to decrease to the minimum value again with a large probability. While the physical and probability considerations are all satisfactory, there does seem to be lack of an explicit mathematical expression of these fluctuations, especially from the point of view of the usual Boltzmann equation.

Very recently, Fox & Uhlenbeck (1969) have studied the question of fluctuations. On statistical theoretical but more or less *ad hoc* ground, they introduce a random 'force' term in the usual (linearized) Boltzmann equation. On the assumption of general statistical properties concerning this term, they have been able to justify and to extend the earlier theory of Landau & Lifshitz (1959) in which terms representing fluctuations are added in an heuristic manner to the hydrodynamical equations.

Since 1946, the theory of the formulation of a kinetic equation (a generalized Boltzmann equation) on the basis of the Liouville equation has been developed by many authors (Bogoliubov, 1946; Born & Green, 1946, 1947; Kirkwood, 1946, 1947; Yvon, 1935). The so-called B–B–G–K–Y hierarchy of equations have been treated by two methods, namely, the 'functional' method of Bogoliubov which is similar to and an extension of the Chapman–Enskog method of solving the Boltzmann equation, and the 'multiple-time-scale' method suggested by Krylov and Bogoliubov (1947) for non-linear mechanics and applied by Frieman (1963) and Sandri (1963) to the kinetic theory problem. It has been found in both methods that under certain assumptions about the initial conditions (on the many-particle correlations), one obtains a kinetic equation which reduces to the usual Boltzmann equation upon making some approximations. This generalized Boltzmann equation also contains no fluctuations, but leads to a monotonic approach to thermodynamic equilibrium.

The purpose of the present work is to obtain a kinetic equation with fluctuations from the B–B–G–K–Y hierarchy of equations of the Liouville equation, with the method of 'multiple-time-scale', on consideration of the lack of definite knowledge about the initial microscopic state of the gas.

2. *Kinetic Equation with Fluctuations*

We start with the Liouville equation

$$\frac{\partial D}{\partial t} = \{\mathscr{H}, D\} \tag{2.1}$$

where $\mathscr{H}$ is the Hamiltonian of the system (N particles, without external fields), D the density (probability) in Γ space,

$$\mathscr{H} = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{1 \leq i<j}^{N} \phi(|\mathbf{r}_i - \mathbf{r}_j|)$$

$$\int \cdots \int D(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N; \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)\, d\mathbf{X}_1 \ldots d\mathbf{X}_N = 1 \tag{2.2a}$$

$$d\mathbf{X}_i \equiv d\mathbf{q}_i\, d\mathbf{p}_i$$

and the curly brackets denote the Poisson brackets. Let F_s be defined by

$$F_s(\mathbf{X}_1, \ldots, \mathbf{X}_s; t) \equiv V^s \int \cdots \int D \, d\mathbf{X}_{s+1} \ldots d\mathbf{X}_N \tag{2.2b}$$

so that F_s has the dimension (momentum)$^{-3s}$. V is the volume of the system.

$$\frac{N}{V} \int F_1(\mathbf{q}, \mathbf{p}; t) \, d\mathbf{p} = n(\mathbf{q}, t) \tag{2.2c}$$

is the number density of particles at $\mathbf{q}$ and t. From the Liouville equation, one obtains the B–B–G–K–Y hierarchy of equations (Wu, 1966)

$$\frac{\partial F_1}{\partial t} + K_1 F_1(1) = n_0 L_1(1;2) F_2(1,2) \tag{2.3}$$

$$\frac{\partial F_2}{\partial t} + \kappa_2 F_2 = n_0 L_2(1,2;3) F_3(1,2,3) \qquad \text{etc.,} \tag{2.4}$$

where

$$\left.\begin{aligned} n_0 &\equiv \frac{N}{V} \\ K_1 &\equiv \frac{1}{m} \mathbf{p}_1 \cdot \nabla_1, \qquad 1 \text{ is the particle index,} \end{aligned}\right\} \tag{2.5a}$$

$$\left.\begin{aligned} L_1(1;j) &\equiv \iint d\mathbf{q}_j \, d\mathbf{p}_j \nabla_1 \phi_{1j} \cdot \frac{\partial}{\partial \mathbf{p}_1} \\ L_2(1,2;j) &\equiv L_1(1;j) + L_1(2;j) \\ \kappa_2 &= \sum_{i=1}^{2} \frac{1}{m} \mathbf{p}_i \cdot \nabla_i - \sum_{i=1}^{2} \nabla_i \phi_{ij} \cdot \frac{\partial}{\partial \mathbf{p}_i} \ldots \text{etc.} \end{aligned}\right\} \tag{2.5b}$$

The formal solution of (2.4) is

$$F_2(1,2;t) = e^{-\kappa_2 t} F_2(1,2;0) + n_0 e^{-\kappa_2 t} \int_0^t d\lambda \, e^{\kappa_2 \lambda} L_2(1,2;j) F_3(1,2,j;\lambda) \tag{2.6}$$

and (2.3) becomes

$$\frac{\partial F_1}{\partial t} + K_1 F_1 = n_0 L_1(1;j) \left\{ e^{-\kappa_2 t} F_2(1,2;0) + n_0 e^{-\kappa_2 t} \int_0^t d\lambda \, e^{\kappa_2 \lambda} L_2(1,2;j) F_3(1,2,j;\lambda) \right\} \tag{2.7}$$

$F_3(1,2,3;t)$ is given by the next equation in the hierarchy (2.3, 2.4) in terms of $F_4(1,2,3,4;t)$, etc. If these functions are known from their initial conditions, then the 'kinetic equation' (2.7) in principle completely determines $F_1(\mathbf{q},\mathbf{p};t)$ without any random fluctuations.

Now in the spirit of statistical theories, we do not possess, nor are we interested in, a knowledge of the microscopic state of a system (a gas of N particles). We can only specify a macroscopic state, in terms of macroscopic properties. But any given macroscopic state corresponds to a large number of microscopic states. For example, a knowledge of the macroscopic quantities

Particle density: $$n(\mathbf{r},t) = n_0 \int F_1\, d\mathbf{p} \tag{2.8a}$$

Mean Flow velocity: $$\mathbf{u}(\mathbf{r},t) = n_0 \int \frac{\mathbf{p}}{m} F_1\, d\mathbf{p} \tag{2.8b}$$

Mean energy density: $$E(\mathbf{r},t) = n_0 \int \frac{p^2}{2m} F_1\, d\mathbf{p} \tag{2.8c}$$

leaves the function $F_1(\mathbf{q},\mathbf{p};t)$ still unspecified. In this sense, we shall regard the initial values of the F_s as random functions.

We shall express the kinetic equation (2.7) in a form which is closer to the usual Boltzmann equation. Let τ_0, τ_1, τ_2 be the characteristic times defined by the time of a collision, the time between two successive collisions and the time, say, for sound to traverse a macroscopic distance, namely,

$$\begin{aligned} \tau_0 &= \frac{r_0}{u} \\ \tau_1 &= \frac{\lambda}{u} \\ \tau_2 &= \frac{L}{u_s} \end{aligned} \tag{2.9}$$

where r_0, λ, L are respectively the range of intermolecular interaction, the mean free path and a macroscopic length, and u is the mean thermal speed of the molecules, and u_s is the speed of sound in the system. For ordinary temperatures and densities,

$$\tau_0 : \tau_1 : \tau_2 \cong 10^{-12} : 10^{-8} : 10^{-4} \tag{2.10}$$

In a gas, the state evolves in time in a complicated way. Let the time t be replaced by the three independent time variables

$$t = t(\tau_0, \tau_1, \tau_2) \tag{2.11a}$$

with

$$\frac{\partial \tau_n}{\partial t} = \varepsilon^n; \qquad n = 0, 1, 2 \tag{2.11b}$$

where ε is a parameter $\ll 1$ ($\varepsilon \simeq 10^{-4}$ in equation (2.10)).

These three time scales are independent since their origins are completely arbitrary. In fact even when a gas has come to thermal equilibrium, processes involving collisions persist. These collisions lead to fluctuations (in n, $\mathbf{u}$ and E in (2.8)) in the time scale τ_1.

We shall expand all functions $F_1, F_2, F_3, \ldots$ in powers of ε

$$
\begin{aligned}
F_1 &= \sum_n \varepsilon^n F_1^{(n)}(\tau_0, \tau_1, \ldots) \\
F_s &= \sum_n \varepsilon^n F_s^{(n)}(\tau_0, \tau_1, \ldots); \qquad s = 2, 3, \ldots, N
\end{aligned} \tag{2.12}
$$

From (2.11), we have

$$\frac{d}{dt} = \frac{\partial}{\partial \tau_0} + \varepsilon \frac{\partial}{\partial \tau_1} + \varepsilon^2 \frac{\partial}{\partial \tau_2} + \cdots \tag{2.13}$$

Equations (2.3), (2.4) become, on dropping the subscript 1 from F_1 (Wu, 1966),

$$\frac{\partial F^{(0)}}{\partial \tau_0} + K_1 F^{(0)} = 0 \tag{2.14}$$

$$\frac{\partial F^{(0)}}{\partial \tau_1} + \frac{\partial F^{(1)}}{\partial \tau_0} + K_1 F^{(1)} = n_0 L_1(1; j) F_2^{(0)}(1, j) \tag{2.15}$$

$$\frac{\partial F_2^{(0)}}{\partial \tau_0} + \kappa_2 F_2^{(0)} = 0 \tag{2.16}$$

$$\frac{\partial F_2^{(0)}}{\partial \tau_1} + \frac{\partial F_2^{(1)}}{\partial \tau_0} + \kappa_2 F_2^{(1)} = n_0 L_2(1, 2; j) F_3^{(0)}(1, 2, j) \tag{2.17}$$

$$\frac{\partial F_s^{(0)}}{\partial \tau_0} + \kappa_s F_s^{(0)} = 0 \tag{2.18}$$

where κ_s is obtained from κ_2 in (2.5) by extending the summation to $j > i =$ from 1 to s.

Let us make the Ansatz

$$
\begin{aligned}
F_2^{(0)}(1, 2; \tau_0, \tau_1, \ldots) = {} & F^{(0)}(1; \tau_0, \tau_1, \ldots) F^{(0)}(2; \tau_0, \tau_1, \ldots) \\
& + G^{(0)}(1, 2; \tau_0, \tau_1, \ldots)
\end{aligned} \tag{2.19}
$$

The solution of (2.4) is

$$
\begin{aligned}
F^{(0)}(1; \tau_0, \tau_1, \ldots) &= e^{-K_1 \tau_0} F^{(0)}(1; \tau_0 = 0, \tau_1, \ldots) \\
&\equiv Z_{-\tau_0}^{(1)} F^{(0)}(1; 0, \tau_1, \ldots)
\end{aligned} \tag{2.20}
$$

or

$$F^{(0)}(1; 0, \tau_1) = Z_{\tau_0}^{(1)}(1) F^{(0)}(1; \tau_0, \tau_1, \ldots) \tag{2.20a}$$

The solution of (2.16) is

$$
\begin{aligned}
F_2^{(0)}(1, 2; \tau_0, \tau_1, \ldots) &= e^{-\kappa_2 \tau_0} F_1^{(0)}(1, 2; 0, \tau_1, \ldots) \\
&\equiv Z_{-\tau_0}^{(2)}(1, 2) \left[\prod_{i=1}^{2} Z_{\tau_0}^{(1)}(i) F^{(0)}(i; \tau_0, \tau_1, \ldots) + G^{(0)}(1, 2; 0, \tau_1, \ldots) \right]
\end{aligned} \tag{2.21}
$$

It is important to note here that the two-particle correlation function $G^{(0)}(1, 2; \tau_0, \tau_1, \ldots)$ at $\tau_0 = 0$ must not be set equal to zero, but $G^{(0)}(1, 2; 0, \tau_1, \ldots)$ must be left as an unknown random function in the $\tau_1, \tau_2, \ldots$ time scales.

If we put (2.21) into (2.15) and set

$$\frac{\partial F^{(0)}}{\partial \tau_1} = n_0 L_1(1,2) Z^{(2)}_{-\infty}\left[\prod_{i=1}^{2} Z^{(1)}_{\infty} F^{(0)}(i) + G^{(0)}(1,2;0,\tau_1,\ldots)\right] \quad (2.22)$$

where

$$\begin{aligned} Z^{(1)}_{\pm\infty}(1) &= \lim_{\tau_0\to\pm\infty} e^{\pm K_1\tau_0} \\ Z^{(2)}_{\pm\infty}(1,2) &= \lim_{\tau_0\to\pm\infty} e^{\pm K_2\tau_0} \end{aligned} \quad (2.22a)$$

then equation (2.15) will approach, as $\tau_0 \to \infty$ (i.e., $\gg \tau_0$),

$$\frac{\partial F^{(1)}}{\partial \tau_0} + K_1 F^{(1)} \to 0 \quad (2.23)$$

and $F^{(1)}$ does not increase secularly in the τ_0 scale. On combining (2.14) and (2.22), and using (2.11) we obtain

$$\left(\frac{\partial}{\partial t} + K_1\right) F^{(0)}(1)$$
$$= n_0 L_1 Z^{(2)}_{-\infty} \prod_{i=1}^{2} Z^{(1)}_{\infty}(i) F^{(0)}(i) + n_0 L_1(1;2) G^{(0)}(1,2;\tau_1,\ldots) \quad (2.24)$$

where

$$G^{(0)}(1,2;\tau_1,\ldots) \equiv Z^{(2)}_{-\infty} G^{(0)}(1,2;0,\tau_1,\ldots) \quad (2.25)$$

$G^{(0)}(1,2;\tau_1,\ldots)$ is thus the value of the two-particle correlation function traced back, in accordance with the equations of motion of the two particles governed by the Hamiltonian $\mathscr{H}_2$ in (2.2). If we impose the condition that

$$G^{(0)}(1,2;\tau_1,\ldots) = 0 \quad (2.26)$$

equation (2.24) is then the generalized Boltzmann equation obtained by Bogoliubov (Bogoliubov, 1946; Wu, 1966),

$$\left(\frac{\partial}{\partial t} + K_1\right) F^{(0)} = n_0 L_1(1;2) Z^{(2)}_{-\infty} \prod_{i=1}^{2} Z^{(1)}_{\infty} F^{(0)}(i) \quad (2.27)$$

which reduces to the usual Boltzmann equation if the inhomogeneity of $F^{(0)}$ within a region of the size of the range r_0 of intermolecular interaction is neglected. In this case, there is no fluctuation.

The initial condition (2.26), however, is too strong a condition and must be examined more closely. We must remember that the time-variables τ_0 and τ_1 are independent in the sense that, if we regard the time t as a linear sequence, at any instant t, there are imbedded in t the variables τ_0, τ_1 whose origins are independent and arbitrary. In an interval of order τ_1 there will be of the order $1/\varepsilon$ intervals of τ_0, and $1/\varepsilon$ independent origins of these $1/\varepsilon$ intervals of τ_0 over each of which the time scale τ_0 spans. The condition (2.26) amounts then to setting the correlation $G^{(0)}(1,2;0,\tau_1,\ldots) = 0$ independently of t in the $\tau_1, \tau_2, \ldots$ scales. The condition is of course mathematically permissible; but it defines a time-arrow by an initial instant, or a 'past', at which there are no correlations so that the system evolves in that direction of time in which the particles become correlated by virtue of their interactions. The result is the time-irreversible Boltzmann equation. From the physical point of view, the assumption of (2.26), or other definite knowledge about the initial condition of the gas, is not consistent with the spirit of the theory of macroscopic description of a gas.

In view of the discussions on the statistical point of view following (2.7) above, we shall regard $G^{(0)}(\tau_1, \ldots)$ in (2.25) as an experimentally unknown random function in the τ_1 time scale. Equation (2.24) is then a 'Boltzmann equation' with a random force term, namely,

$$\left(\frac{\partial}{\partial t} + K_1\right) F^{(0)}(1) = n_0 L_1(1;2) Z^{(2)}_{-\infty} \prod_{i=1}^{2} Z^{(1)}_{\infty} F^{(0)}(i) + \tilde{C}(1;t) \quad (2.28)$$

where

$$\tilde{C}(\mathbf{q}_1, \mathbf{p}_1; t) \equiv n_0 \iint d\mathbf{q}_2\, d\mathbf{p}_2\, \nabla_1 \phi(|\mathbf{r}_{12}|) \cdot \frac{\partial}{\partial \mathbf{p}_1} G^{(0)}(1,2;t) \quad (2.29)$$

is a random function since $G^{(0)}(1,2;t)$ is a random function.

On writing

$$F^{(0)} = f(1+\varphi) \equiv f + \tilde{f} \quad (2.30)$$

where f is a solution of the Boltzmann equation (2.27), and for 'small' fluctuations

$$|\tilde{f}| \ll f \quad (2.31)$$

one obtains the linearized kinetic equation

$$\left(\frac{\partial}{\partial t} + K_1\right)\tilde{f}$$
$$= n_0 L_1 Z^{(2)}_{-\infty}[Z^{(1)}_{\infty} f(1) Z^{(1)}_{\infty} \tilde{f}(2) + Z^{(1)}_{\infty} f(2) Z^{(1)}_{\infty} \tilde{f}(1)] + \tilde{C}(1;t) \quad (2.32)$$

If we are interested in the fluctuations from the equilibrium state, i.e.,

$$f = f_{\text{equil}} \equiv f_{\text{eq}}(p) \quad (2.33)$$

this linearized equation becomes

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla\right) \varphi(\mathbf{q}, \mathbf{p}; t) = \iint dp_1\, d\Omega\, g\sigma(g,\theta) f_{\text{eq}}(p_1) [\varphi(\mathbf{q}', \mathbf{p}'; t) \\ + \varphi(\mathbf{q}', \mathbf{p}_1'; t) - \varphi(\mathbf{q}, \mathbf{p}; t) - \varphi(\mathbf{q}, \mathbf{p}_1; t)] \\ + \tilde{C}(\mathbf{q}, \mathbf{p}; t) \quad (2.34)\end{aligned}$$

where $g = (1/m)|\mathbf{p}_1 - \mathbf{p}|$, σ is the cross-section of the two-body collision, $d\Omega$ is the solid angle of the scattering, and $\mathbf{p}'$, $\mathbf{p}_1'$ are the momenta of the two particles such that after collision at $\mathbf{q}$ they become $\mathbf{p}$, $\mathbf{p}_1$ respectively. Equation (2.34) without the $\tilde{C}$ term is the usual linearized Boltzmann equation in the Chapman–Engskog solution of the Boltzmann equation. Equation (2.34) is just that recently obtained by Fox (1969) who introduces a random force term in the usual linearized Boltzmann equation on statistical considerations. The approach from the Liouville equation in the present work thus has furnished a source for a fluctuation term in the Boltzmann equation.

3. *The H-Theorem*

The original statement of the H theorem (1.1) by Boltzmann on the basis of the Boltzmann equation has been criticized by Loschmidt and Zermelo. These criticisms were based on the assumption that the evolution of the H function is determined by dynamical laws. In fact, if the kinetic equation, written in the form

$$\frac{\partial F}{\partial t} = I(F) \tag{3.1}$$

is time-reversal invariant, i.e., if upon time reversal

$$\begin{gathered} t \to -\tau \\ \mathbf{p} \to -\mathbf{p} \\ F(\mathbf{r}, \mathbf{p}; t) \to F(\mathbf{r}, -\mathbf{p}; -\tau) \equiv \tilde{F}(\mathbf{r}, \mathbf{p}; \tau) \end{gathered} \tag{3.2}$$

(3.1) goes into

$$\frac{\partial \tilde{F}}{\partial \tau} = I(\tilde{F}), \tag{3.3}$$

then it can readily be seen that

$$\frac{dH}{dt} = 0. \tag{3.4}$$

The Boltzmann equation containing the Stosszahlansatz is however not a consequence of dynamical laws and is not invariant under time reversal. That the Boltzmann equation leads to (1.1) is therefore not subject to the objections of Loschmidt and Zermelo. Boltzmann has subsequently reformulated the H theorem by giving it a probability interpretation. The collision integral is taken to give the probable decrease of H whenever H is above the asymptotic minimum given by $dH/dt = 0$. The H function for one individual system may rise, but the average of the H function of a large number of systems is to be described by the H theorem (1.1). This formulation of the H theorem has been discussed by Boltzmann (1898), P. & T. Ehrenfest (1911) and ter Haar (1954). It remains, as noted in the Introduction above, to have an explicit theory for describing the decrease of the H function on the average but with fluctuations.

We shall first show that the fine-grained time behavior (in the τ_0 time scale) of the system as described by the Liouville equation does not yield fluctuations. Let us take equation (2.3), and write

$$F(\tau_0, \tau_1, \ldots) = f(\tau_1, \ldots) + \psi(\tau_0, \tau_1, \ldots) \tag{3.5}$$

where f is the 'slowly' varying part, i.e.,

$$f(\tau_1,\ldots)=\lim_{T\to\infty}\frac{1}{T}\int_0^T F(\tau_0,\tau_1,\ldots)\,d\tau_0 \tag{3.6}$$

or

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \psi(\tau_0,\tau_1,\ldots)\,d\tau_0=0. \tag{3.6a}$$

Similarly, for the G in

$$F_2(1,2)=F(1)F(2)+G(1,2) \tag{3.7}$$

we write

$$G(\tau_0,\tau_1,\ldots)=g(\tau_1,\ldots)+\chi(\tau_0,\tau_1,\ldots) \tag{3.8}$$

where

$$g(\tau_1,\ldots)=\lim_{T\to\infty}\frac{1}{T}\int_0^T G(\tau_0,\tau_1,\ldots)\,d\tau_0. \tag{3.9}$$

On making expansions of F, G as in (2.12), and if we assume the initial condition

$$G^{(0)}(\tau_0=0,\tau_1,\ldots)=0 \tag{3.10}$$

we obtain for $f(\tau_1,\ldots)$ the generalized Boltzmann equation (2.27). If we define the coarse-grained time H function by

$$\bar{H}=\iint f\ln f\,d\mathbf{q}\,d\mathbf{p} \tag{3.11}$$

then we obtain

$$\frac{d\bar{H}}{dt}=\frac{dH}{d\tau_1}\leqslant 0 \tag{3.12}$$

which is the H theorem in the usual form from the Boltzmann equation. The fine-grained time H function is defined by F of (3.4)

$$H=\iint F\ln F\,d\mathbf{q}\,d\mathbf{p} \tag{3.13}$$

in the τ_0 time scale. In this time scale, F is given by the time-reversal invariant equation (2.3), and it follows from (3.1)–(3.4),

$$\frac{dH}{d\tau_0}=0. \tag{3.14}$$

We summarize the above result by saying that if we assume the initial condition (3.10), then the Liouville equation leads to an H function which decreases monotonically in the coarse-grained τ_1 time scale, but in the fine-grained τ_0 time scale remains constant. There are no 'fluctuations'.

Let us now use the kinetic equation (2.28) with the random term $\tilde{C}(\mathbf{q},\mathbf{p};t)$. On putting (2.30) into (3.13), we obtain

$$\frac{dH}{dt}=\frac{d}{dt}\iint F\ln F\,d\mathbf{q}\,d\mathbf{p}$$

$$=\iint d\mathbf{q}\,d\mathbf{p}\left[(1+\ln f)\frac{\partial f}{\partial t}+(1+\ln f)\frac{\partial \tilde{f}}{\partial t}+\ln(1+\varphi)\frac{\partial F}{\partial t}\right] \quad (3.15)$$

The first term is the dH/dt as given by the usual Boltzmann equation and is $(dH/dt)_{\text{Boltz}} \leqslant 0$. The other two terms contain the fluctuations $\tilde{f}$ determined by the random function $\tilde{C}(\mathbf{q},\mathbf{p};t)$ in (2.32) and may have all values $\gtreqless 0$. It is to be noted that these fluctuations in H are in the τ_1 (mean free) time scale. These fluctuations in H correspond to fluctuations in the entropy in the approach of a system to equilibrium.

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